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A PRELIMINARY INVESTIGATION OF THE MOTION OF A LONG, FLEXIBLE WIRE IN ORBIT

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SUMMARY

This memorandum is concerned with the motion of a long, flexible wire in orbit. The wire is approximated by n point masses joined by massless rigid rods.

Under the assumption of an inverse square gravitational field, the general equations of motion of the n masses are developed. Then, with all external forces except gravity neglected, and with the motion of the wire restricted to the orbital plane, two specific cases are examined. In the first case the wire spins about its center of mass, after having been initially extended in some fashion. The second case examines the motion of an extended wire which, initially, is approximately aligned with the vertical.

The analysis indicates that, in the absence of internal dissipative forces, the straight-line configuration of the spinning wire is not stable, since a parametric resonance condition exists. In the second case the wire will oscillate about the vertical if the initial disturbances are small.

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SYMBOLS

$\bar{r}_o$	vector position of the center of mass with respect to an earth-centered inertial reference frame
$\bar{\rho}_i$	vector position of the i-th particle with respect to the system center of mass
$\bar{F}_{gi}$	external gravitational force acting on i-th particle
$\bar{F}_{si}$	internal system forces acting on the i-th particle
m_i	mass of the i-th particle
M	total mass of system of particles
$\left.\begin{matrix}\eta\\ \zeta\end{matrix}\right\}$	coordinates located at center of mass, rotating with respect to the local vertical
α	angle between local vertical and the η axis (see Fig. 2)
l	total length of the system of particles
$F_{Ti,i+1}$	tension force between the i-th and i-th plus 1 particles
l_p	the inter-particle distance
Ω	the initial angular rate, $\dot{\alpha}(0)$, of the η-ζ axis

I. INTRODUCTION

The use of satellites for various communication purposes has been under study at RAND for some time. One proposed type of satellite consists of a long wire in orbit. The wire, which serves as an antenna, would be kept in tension by spinning it about its center of mass. This device would provide intermittent communication between a sending and a receiving station.

A wire is attractive for this application since, being flexible, a long antenna may be contained in a relatively compact package for the ascent phase.

The relationship between the orbital altitude, the spin rate, the frequency of the transmitted radiation and the characteristics of the antenna have been examined in Ref. 1, while in Ref. 2 the mechanical design requirements of a rotating antenna have been considered.

System studies indicate that an interesting design is a wire approximately 2000 ft long, orbiting at an altitude of from 1500 to 2000 miles. For these conditions, the wire should not deviate from a straight line by more than $\lambda/8$, where λ is the wavelength of the incident radiation.(1) For typical values of λ, the permissible transverse deflection of the cable is of the order of a few centimeters!(1) Various design considerations lead to a spin rate, with respect to the local vertical, of approximately one radian per second.(1,2) This research memorandum is concerned with the stability of a spinning wire when subjected to alternating tension and compression forces due to the gravitational gradient.

After the packaged wire has been placed into orbit, the wire must be unwound or extended in some fashion and then accelerated to the desired spin rate. Once the wire is spinning in the orbital plane, it may oscillate due

to the influence of the gravitational gradient and other small perturbing forces. Thus the question arises, whether the straight-line configuration of the antenna is stable.

In this research memorandum, the spin-up phase is not considered. It is assumed that initially the wire is spinning in the orbital plane, and that the transverse deflections are very small compared to the length of the antenna. Since a wire is a continuous, flexible body, its transverse motion can be expressed in terms of partial differential equations. However, the gross characteristics of a flexible wire may be approximated by a number of point masses joined by rigid rods without mass. It is hoped that by examining a simple model of this sort, any obvious stability problems will be revealed. Furthermore, insight should be gained to the mathematical and physical approximations that can be made in an analysis of a continuous wire.

II. DERIVATION OF THE EQUATIONS OF MOTION

In Fig. 1, the variables used to define the motion of the center of mass are indicated, while Fig. 2 shows the coordinates used to define the positions of the n point masses. Only the planar case will be considered.

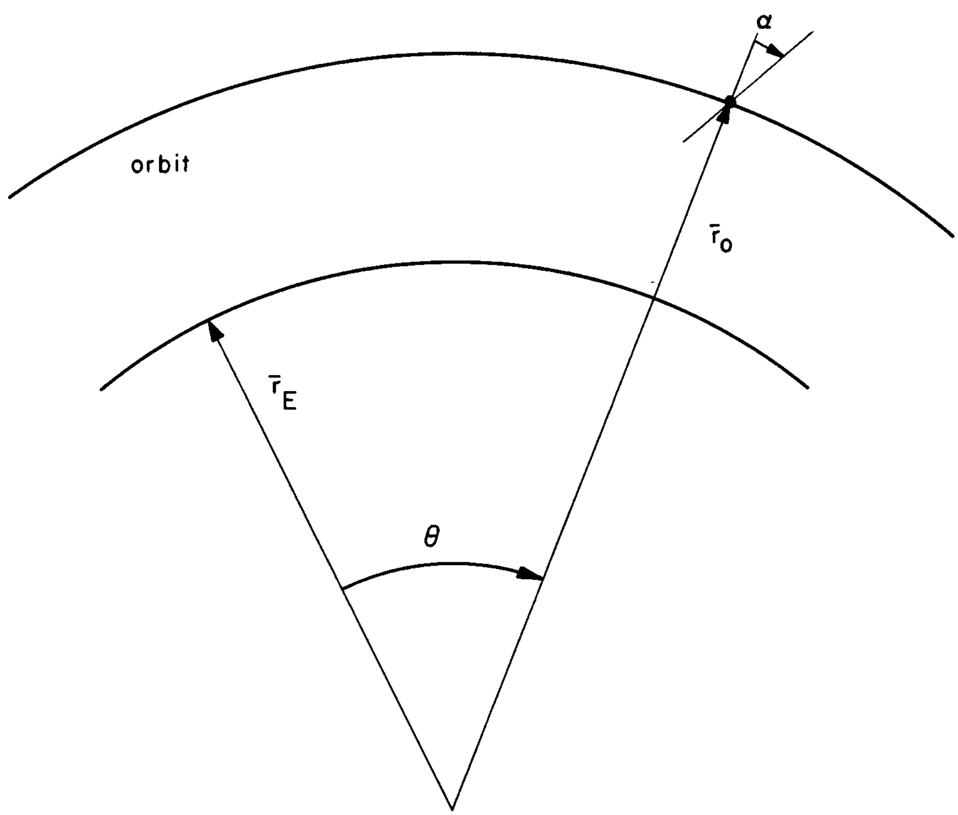

Fig. 1 — Center of mass coordinates

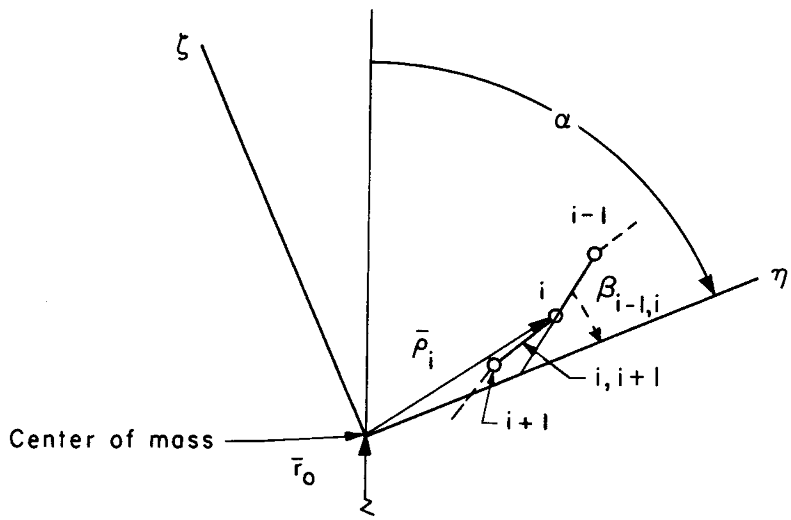

Fig. 2 — Particle coordinates

The acceleration of the i-th mass with respect to an inertial frame is

$$m_i \left[(\ddot{\bar{r}}_o)_I + (\ddot{\bar{\rho}}_i)_I \right] = \bar{F}_{gi} + \bar{F}_{si} \tag{1}$$

where $\bar{F}_{gi}$ is the external gravitational force acting on the i-th particle and $\bar{F}_{si}$ represents all of the forces internal to the system which act on the i-th particle. From the definition of the center of mass

$$\sum_1^n m_i \bar{\rho}_i = 0 \tag{2}$$

Noting that $\sum_1^n \bar{F}_{si}$ must be zero, Eq. (1) may be written as follows:

$$M(\ddot{\bar{r}}_o)_I = \sum_1^n \bar{F}_{gi} \tag{3a}$$

$$m_i(\ddot{\bar{\rho}}_i)_I = -\frac{m_i}{M} \sum_1^n \bar{F}_{gi} + \bar{F}_{gi} + \bar{F}_{si} \tag{3b}$$

Equation (3a) defines the motion of the center of mass of the wire, while Eq. (3b) defines the motion of the i-th particle with respect to the center of mass as observed from an inertial reference frame. When the motion of the i-th particle is seen from the rotating η-ζ coordinate system, Eq. (3b) has the following form:

$$m_i \left[(\ddot{\bar{\rho}}_i)_{ms} + 2\bar{\omega}_{ms} \times (\dot{\bar{\rho}}_i)_{ms} + (\dot{\bar{\omega}}_{ms})_{ms} \times \bar{\rho}_i + \bar{\omega}_{ms} \times (\bar{\omega}_{ms} \times \bar{\rho}_i) \right] = - \frac{m_i}{M} \sum_1^n \bar{F}_{gi} + \bar{F}_{gi} + \bar{F}_{si} \tag{4}$$

where the subscript, ms, indicates that vector differentiation is with respect to the rotating system, and $\bar{\omega}_{ms}$ is the angular velocity of the η-ζ system with respect to the inertial frame. Writing Eq. (4) in terms of its scalar components yields

$$m_i \left[\ddot{\zeta}_i - 2\dot{\eta}_i (\dot{\alpha} + \dot{\theta}) - \eta_i(\ddot{\alpha} + \ddot{\theta}) - \zeta_i (\dot{\alpha} + \dot{\theta})^2 \right] = - \frac{m_i}{M} \sum_1^n F_{g\zeta i} + F_{g\zeta i} + F_{s\zeta i} \tag{5a}$$

$$m_i \left[\ddot{\eta}_i + 2\dot{\zeta}_i (\dot{\alpha} + \dot{\theta}) + \zeta_i(\ddot{\alpha} + \ddot{\theta}) - \eta_i (\dot{\alpha} + \dot{\theta})^2 \right] = - \frac{m_i}{M} \sum_1^n F_{g\eta i} + F_{g\eta i} + F_{s\eta i} \tag{5b}$$

Thus for n particles there are 2n equations identical in form to Eq. (5). However, all of the 2n equations are not independent, because of center of mass considerations, and because the total length, l, of the 'wire' constrains

the values that the η's and ζ's may assume. From Eq. (2)

$$\sum_{1}^{n} m_i \eta_i = 0 \tag{6a}$$

$$\sum_{1}^{n} m_i \zeta_i = 0 \tag{6b}$$

The length, l, is the sum of the distances between the individual particles. From Fig. 2

$$\sin \beta_{i,i+1} = \frac{\zeta_i - \zeta_{i+1}}{l_{i,i+1}} \tag{7a}$$

$$\cos \beta_{i,i+1} = \frac{\eta_i - \eta_{i+1}}{l_{i,i+1}} \tag{7b}$$

where i runs from 1 to n-1.

Thus the internal force components acting on the i-th particle may be written as follows:

$$F_{s\eta i} = F_{Ti,i-1} \cos \beta_{i-1,i} - F_{Ti,i+1} \cos \beta_{i,i+1} \tag{8a}$$

$$F_{s\zeta i} = F_{Ti,i-1} \sin \beta_{i-1,i} - F_{Ti,i+1} \sin \beta_{i,i+1} \tag{8b}$$

where $F_{Ti,i+1}$ is a tension force, and $F_{Ti,i+1} = F_{T,i+1,i}$. The tension forces are unknown forces of constraint which arise due to the accelerations of the connected particles.

Introducing Eqs. (7) and (8) into Eqs. (5a) and (5b) yields:

$$m_i \left[\ddot{\zeta}_i - 2\dot{\eta}_i (\dot{\alpha} + \dot{\theta}) - \eta_i (\ddot{\alpha} + \ddot{\theta}) - \zeta_i (\dot{\alpha} + \dot{\theta})^2 \right] = -\frac{m_i}{M} \sum_1^n F_{g\zeta i} + F_{g\zeta i}$$

$$+ F_{Ti,i-1} \left(\frac{\zeta_{i-1} - \zeta_i}{l_{i-1,i}} \right) - F_{Ti,i+1} \left(\frac{\zeta_i - \zeta_{i+1}}{l_{i,i+1}} \right) \qquad (9a)$$

$$m_i \left[\ddot{\eta}_i + 2\dot{\zeta}_i (\dot{\alpha} + \dot{\theta}) + \zeta_i (\ddot{\alpha} + \ddot{\theta}) - \eta_i (\dot{\alpha} + \dot{\theta})^2 \right] = -\frac{m_i}{M} \sum_1^n F_{g\eta i} + F_{g\eta i}$$

$$+ F_{Ti,i-1} \left(\frac{\eta_{i-1} - \eta_i}{l_{i-1,i}} \right) - F_{Ti,i+1} \left(\frac{\eta_i - \eta_{i+1}}{l_{i,i+1}} \right) \qquad (9b)$$

Finally, the external gravitational force which acts on the i-th particle may be determined from consideration of Fig. 3.

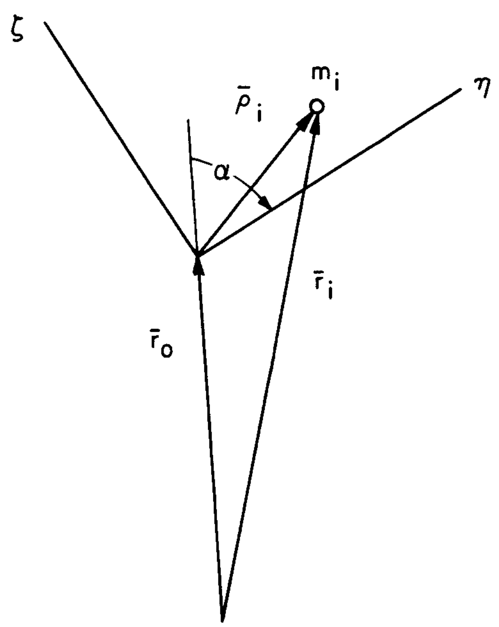

Fig. 3 — Resolution of gravitational force into the $\eta-\zeta$ coordinate system

$$F_{g\zeta i} = - \frac{m_i \; g_o \; r_E^2 \; r_o}{r_i^3} \left[\sin \alpha + \frac{\zeta_i}{r_o} \right] \tag{10a}$$

$$F_{g\eta i} = - \frac{m_i \; g_o \; r_E^2 \; r_o}{r_i^3} \left[\cos \alpha + \frac{\eta_i}{r_o} \right] \tag{10b}$$

The distance from the center of the earth to the i-th particle may be found from the law of cosines:

$$r_i^2 = r_o^2 + \rho_i^2 + 2r_o \left[\eta_i \cos \alpha + \zeta_i \sin \alpha \right] \tag{11}$$

Thus

$$r_i^{-3} \cong r_o^{-3} \left[1 - \frac{3}{2} \left(2 \frac{\eta_i}{r_o} \cos \alpha + 2 \frac{\zeta_i}{r_o} \sin \alpha \right) \right] \tag{12}$$

In the binomial expansion only first-order terms have been retained.*
Introducing Eqs. (10a) and (10b) and (12) into Eqs. (9a) and (9b) yields:

$$\begin{aligned} \ddot{\zeta}_i - 2 \dot{\eta}_i (\dot{\alpha} + \dot{\theta}) - \eta_i (\ddot{\alpha} + \ddot{\theta}) - \zeta_i (\dot{\alpha} + \dot{\theta})^2 & \\ = - \frac{g_o \; r_E^2}{r_o^3} (- 3\eta_i \sin \alpha \cos \alpha - 3\zeta_i \sin^2 \alpha + \zeta_i) & \\ + \frac{F_{Ti,i-1}}{m_i} \left(\frac{\zeta_{i-1} - \zeta_i}{l_{i-1,i}} \right) - \frac{F_{Ti,i+1}}{m_i} \left(\frac{\zeta_i - \zeta_{i+1}}{l_{i,i+1}} \right) & \end{aligned} \tag{13a}$$

$$\begin{aligned} \ddot{\eta}_i + 2\dot{\zeta}_i (\dot{\alpha} + \dot{\theta}) + \zeta_i (\ddot{\alpha} + \ddot{\theta}) - \eta_i (\dot{\alpha} + \dot{\theta})^2 & \\ = - \frac{g_o \; r_E^2}{r_o^3} (- 3\eta_i \cos^2 \alpha - 3\zeta_i \sin \alpha \cos \alpha + \eta_i) & \\ + \frac{F_{Ti,i-1}}{m_i} \left(\frac{\eta_{i-1,i} - \eta_i}{l_{i-1,i}} \right) - \frac{F_{Ti,i+1}}{m_i} \left(\frac{\eta_i - \eta_{i+1}}{l_{i,i+1}} \right) & \end{aligned} \tag{13b}$$

*The influence of higher order terms is considered in Ref. 3.

Thus for n particles interconnected by (n-1) massless, rigid rods there are 2n equations of the form of Eqs. (13a) and (13b) and (3n-1) unknowns. The remaining (n-1) equations are provided by Eqs. (7a) and (7b).

$$l_{i,i+1}^2 = (\eta_i - \eta_{i+1})^2 + (\zeta_i - \zeta_{i+1})^2 \tag{14}$$

$$i = 1, \ldots, n-1$$

The manner in which the η - ζ axes rotate with respect to the local vertical has not yet been considered. One reasonable approach is to select α as a function of time based on the motion of the equivalent rigid body. Rather than prescribing the rotational motion of the η - ζ axes it is also possible to constrain one of the η_i's or ζ_i's by making α an unknown. The latter procedure has been followed in the subsequent development of the equations of motion.

It is assumed that the number of mass particles, n, is odd. If, in the undeflected state, the 'cable' exhibits mirror symmetry with respect to a plane perpendicular to its center, then there will be a particle at that center. By the use of Eq. (14) all of the η's, with the exception of one, may be expressed in terms of the inter-particle lengths and the ζ's. The remaining η will be taken to be that of the center particle and its value will be constrained to be zero. Thus for n particles there are 2n equations, n unknown ζ's, (n-1) unknown tension forces, and α. However, the constraint imposed upon $\eta_{\frac{n+1}{2}}$ does not result in a unique set of η - ζ axes (see Fig. 4). For the symmetric distribution of particles illustrated in Fig. (4), either the η - ζ or the η' - ζ' axes would satisfy the imposed

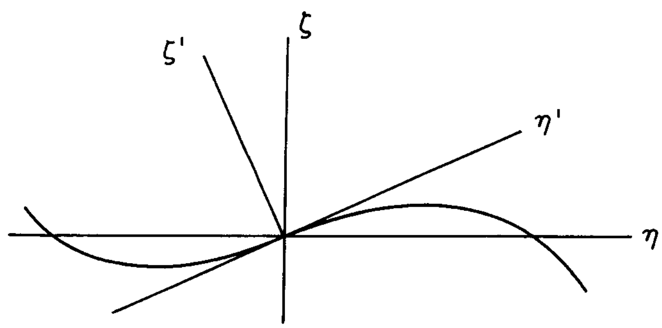

Fig. 4 — Possible $\eta - \zeta$ axes

constraints. For the undeflected cable to always coincide with the η axis, the following condition must be satisfied:

$$\sum_{1}^{\frac{n-1}{2}} m_i \zeta_i - \sum_{\frac{n+3}{2}}^{n} m_i \zeta_i = 0 \tag{15}$$

Equation (15) insures that the η-ζ axes are such that center of mass of one half of the cable has a mirror image.

Finally, in view of the magnitude of the permissible transverse deflection as discussed in Section I, only terms containing the first power of ζ_i will be retained. As a consequence,

$$l_{i,i+1} \cong \eta_i - \eta_{i+1} \tag{16}$$

Equation (16) together with Eq. (6a) indicates that all of the derivatives of the η's must be zero.

Taking into consideration Eqs. (6a), (6b), (15) and (16) it is now possible to obtain from Eq. (13a) an expression for the angular motion of

the η - ζ axes. This is done by summing first over the top half of the cable and then subtracting from that result the summation over the bottom half.

$$
\begin{aligned}
-(\ddot{\alpha} + \ddot{\theta}) \left(\sum_{1}^{j-1} m_i \eta_i \right) &= \frac{3g_o r_E^2}{r_o^3} \sin\alpha \cos\alpha \left(\sum_{1}^{j-1} m_i \eta_i \right) \\
&+ \frac{1}{2} \sum_{1}^{j-1} \left[F_{Ti,i-1} \left(\frac{\zeta_{i-1} - \zeta_i}{l_{i-1,i}} \right) - F_{Ti,i+1} \left(\frac{\zeta_i - \zeta_{i+1}}{l_{i,i+1}} \right) \right] \\
&- \frac{1}{2} \sum_{j+1}^{n} \left[F_{Ti,i-1} \left(\frac{\zeta_{i-1} - \zeta_i}{l_{i-1,i}} \right) - F_{Ti,i+1} \left(\frac{\zeta_i - \zeta_{i+1}}{l_{i,i+1}} \right) \right]
\end{aligned}
\tag{17}
$$

where the index j is identical with $\frac{n+1}{2}$. If the "cable" were rigid, all of the ζ's would be identically zero, and Eq. (17) then reduces to the form that is usually considered when studying gravitational gradient effects.

Following the same summing procedure with respect to Eq. (13b) yields

$$
\begin{aligned}
-(\dot{\alpha} + \dot{\theta})^2 \left(\sum_{1}^{j-1} m_i \eta_1 \right) &= \frac{g_o r_E^2}{r_o^3} (1 - 3\cos^2\alpha) \left(\sum_{1}^{j-1} m_i \eta_i \right) \\
&+ \frac{1}{2} \sum_{1}^{j-1} \left[F_{Ti,i-1} - F_{Ti,i+1} \right] - \frac{1}{2} \sum_{j+1}^{n} \left[F_{Ti,i-1} - F_{Ti,i+1} \right]
\end{aligned}
\tag{18}
$$

If we only consider the motion of the center particle, Eq. (13b) becomes

$$m_j \left[2 \dot{\zeta}_j \, (\dot{\alpha} + \dot{\theta}) + \zeta_j \, (\ddot{\alpha} + \ddot{\theta}) \right] = \frac{3 m_j \, g_o \, r_E^2}{r_o^3} \, \zeta_j \sin \alpha \cos \alpha + F_{Tj,j-1} - F_{Tj,j+1} \tag{19}$$

However, the components of the tension force acting on the j-th particle must be equal in magnitude to the summations of the tension components acting on the remaining particles.

$$\sum_{1}^{j-1} \left[F_{Ti,i-1} - F_{Ti,i+1} \right] = - F_{Tj,j-1} \tag{20a}$$

$$\sum_{j+1}^{n} \left[F_{Ti,i-1} - F_{Ti,i+1} \right] = F_{Tj,j+1} \tag{20b}$$

$$\sum_{1}^{j-1} \left[F_{Ti,i-1} \, (\zeta_{i-1} - \zeta_i) - F_{Ti,i+1} \, (\zeta_i - \zeta_{i+1}) \right] = - F_{Tj,j-1} \, (\zeta_{j-1} - \zeta_j) \tag{20c}$$

$$\sum_{j+1}^{n} \left[F_{Ti,i-1} \, (\zeta_{i-1} - \zeta_i) - F_{Ti,i+1} \, (\zeta_i - \zeta_{i+1}) \right] = F_{Tj,j+1} \, (\zeta_j - \zeta_{j+1}) \tag{20d}$$

Thus the tension forces which act on the j-th particle may be determined in terms of the variables α and ζ, by consideration of Eqs. (18), (19), (20a) and (20b).

$$F_{Tj,j-1} = m_j \left[\dot{\zeta}_j \, (\dot{\alpha} + \dot{\theta}) + \frac{1}{2} \zeta_j \, (\ddot{\alpha} + \ddot{\theta}) \right] + \left(\sum_{1}^{j-1} m_i \, \eta_i \right) (\dot{\alpha} + \dot{\theta})^2$$

$$- \frac{3 g_o \, r_E^2}{2 r_o^3} m_j \, \zeta_j \sin \alpha \cos \alpha - \frac{g_o \, r_E^2}{r_o^3} (1 - 3 \cos^2 \alpha) \left(\sum_{1}^{j-1} m_i \, \eta_i \right) \tag{21a}$$

$$F_{Tj,j+1} = (\dot{\alpha} + \dot{\theta})^2 \left(\sum_{1}^{j-1} m_i \, \eta_i \right) + \frac{3 g_o \, r_E^2}{2 r_o^3} \, m_j \, \zeta_j \sin \alpha \cos \alpha$$

$$- m_j \left[\dot{\zeta}_j \, (\dot{\alpha} + \dot{\theta}) + \frac{1}{2} \zeta_j \, (\ddot{\alpha} + \ddot{\theta}) \right] - \frac{g_o \, r_E^2}{r_o^3} (1 - 3 \cos^2 \alpha) \left(\sum_{1}^{j-1} m_i \, \eta_i \right) \tag{21b}$$

Utilizing Eqs. (20c), (20d), (21a), and (21b), Eq. (17) may be written as follows:

$$\ddot{\alpha} + \ddot{\theta} + \frac{3 g_o \, r_E^2}{r_o^3} \sin \alpha \cos \alpha = \frac{1}{2 l_p} \, (\zeta_{j-1} - \zeta_{j+1}) \left[(\dot{\alpha} + \dot{\theta})^2 - \frac{g_o \, r_E^2}{r_o^3} (1 - 3 \cos^2 \alpha) \right] \tag{22}$$

where l_p is the inter-particle distance, assumed to be a constant for all of the particles.

Introducing Eqs. (20a) and (20b) and (21a) and (21b) into Eq. (13a) yields the following transverse equation of motion for the center particle.

$$\ddot{\zeta}_j + \zeta_j \left\{ \frac{2 \sum_{1}^{j-1} m_i \, \eta_i}{l_p \, m_j} \left[(\dot{\alpha} + \dot{\theta})^2 - \frac{g_o \, r_E^2}{r_o^3} \, (1 - 3 \cos^2 \alpha) \right] - (\dot{\alpha} + \dot{\theta})^2 \right.$$

$$+ \frac{g_o \, r_E^2}{r_o^3} (1\text{-}3 \sin^2 \alpha)\Bigg\} \quad = \quad \frac{\sum_{1}^{j-1} m_i \, \eta_i}{l_p \, m_j} \Bigg\{\Bigg[(\dot{\alpha} + \dot{\theta})^2$$

$$- \frac{g_o \, r_E^2}{r_o^3} \; (1\text{-}3 \cos^2 \alpha)\Bigg] \quad (\zeta_{j-1} + \zeta_{j+1})\Bigg\} \tag{23}$$

We are now able to write the remaining transverse equations of motion in terms of functions of α and the ζ's. The tension force acting on the j-1 particle due to the j-2 particle is, from Eq. (13b)

$$F_{Tj-1,j-2} = F_{Tj-1,j} + m_{j-1} \; \Bigg[2\dot{\zeta}_{\,j-1} \; (\dot{\alpha} + \dot{\theta}) + \zeta_{j-1} \; (\ddot{\alpha} + \ddot{\theta})$$

$$- \; \eta_{j-1} \; (\dot{\alpha} + \dot{\theta})^2 + \frac{g_o \, r_E^2}{r_o^3} \; (\eta_{j-1} - 3\zeta_{j-1} \sin \alpha \cos \alpha - 3\eta_{j-1} \cos^2 \alpha)\Bigg] \tag{24}$$

However, $F_{Tj-1,j}$ is identical with $F_{Tj,j-1}$ of Eq. (21a). Substituting Eqs. (21a), (22) and (24) into Eq. (13a) yields the following expression for the transverse motion of the j-1 particle:

$$\ddot{\zeta}_{\,j-1} + \zeta_{\,j-1} \; \Bigg\{ \Bigg[(\dot{\alpha} + \dot{\theta})^2 - \frac{g_o \, r_E^2}{r_o^3} (1\text{-}3 \cos^2 \alpha)\Bigg] \Bigg[\frac{2 \sum_{1}^{j-1} m_i \, \eta_i}{m_{j-1} \, l_p} - \frac{3 \, \eta_{j-1}}{2 l_p} \Bigg]$$

$$- \; (\dot{\alpha} + \dot{\theta})^2 + \frac{g_o \, r_E^2}{r_o^3} \; (1\text{-}3 \sin^2 \alpha)\Bigg\} \quad = \quad \Bigg[(\dot{\alpha} + \dot{\theta})^2 \tag{25}$$

$$- \frac{g_o \, r_E^2}{r_o^3} (1-3 \cos^2 \alpha) \Bigg] \left[\frac{\sum_1^{j-1} m_i \, \eta_i}{m_{j-1} \, l_p} (\zeta_{j-2} + \zeta_j) - \frac{\eta_{j-1}}{2 \, l_p} (\zeta_{j+1} + 2\zeta_{j-2}) \right]$$

The procedure that has been followed in obtaining Eq. (25) may be applied in turn to the j-2 particle, etc. Because of the symmetry that exists, the equation of motion of any one of the particles comprising the lower half of the 'wire' may be determined from the corresponding upper-half equation by substitution of indices, and noting that

$$\eta_{j-1} = - \eta_{j+1}, \text{ etc.} \tag{26}$$

From center of mass considerations

$$m_1 \, \zeta_1 = - \frac{1}{2} m_j \, \zeta_j - \left[m_2 \, \zeta_2 + \dots + m_{j-1} \, \zeta_{j-1} \right] \tag{27a}$$

$$m_n \, \zeta_n = - \frac{1}{2} m_j \, \zeta_j - \left[m_{j+1} \, \zeta_{j+1} + \dots + m_{n-1} \, \zeta_{n-1} \right] \tag{27b}$$

Thus after the elimination of the tension force expressions, there are (n+1) equations to be solved. These are the (n-2)ζ equations, the α equation and the two equations of the center of mass motion. Of these equations, (n-1) are coupled.

III. SOLUTION OF THE EQUATIONS OF MOTION

From Fig. 1 and Eqs. (3a) and (12), the orbital equations of motion are:

$$\ddot{r}_o - r_o \dot{\theta}^2 = - \frac{g_o r_E^2}{r_o^2} \tag{28a}$$

$$\frac{d}{dt} (r_o^2 \dot{\theta}) = 0 \tag{28b}$$

Thus the rotational and vibrational degrees of freedom do not influence the center-of-mass motion.* It will be assumed that the orbit is circular, and that $\dot{\theta}$ equals ω_c, a constant.

The equations to be solved are:

$$\ddot{\alpha} + 3\omega_c^2 \sin \alpha \cos \alpha = \frac{s_1}{2l_p} (\zeta_{j-1} - \zeta_{j+1}) \tag{29a}$$

$$\ddot{\zeta}_{j\mp k} + \zeta_{j\mp k} \left[- s_2 + s_1 \left(\frac{2Q + m_{j-k} \eta_{j-k}}{m_{j-k} l_p}\right) \right]$$

$$= s_1 \left[\frac{\eta_{j\mp k}}{2l_p} (\zeta_{j-1} - \zeta_{j+1}) + \left(\frac{Q + m_{j-k} \eta_{j-k}}{m_{j-k} l_p}\right) \zeta_{j\mp(k-1)} \right.$$

$$\left. + \frac{Q \zeta_{j\mp(k+1)}}{m_{j-k} l_p} \right] \tag{29b}$$

---

*This is due to the fact that in the binomial expansion of r_i^{-3}, only first-order terms have been retained. (See Ref. 3.)

where
$$k = 0, 1, \ldots, (j-2)$$

$$s_1 = (\dot{\alpha} + \omega_c)^2 - \omega_c^2 (1-3 \cos^2 \alpha)$$

$$s_2 = (\dot{\alpha} + \omega_c)^2 - \omega_c^2 (1-3 \sin^2 \alpha)$$

$$Q = \sum_{1}^{j-1} m_i \eta_i - \sum_{j-1}^{j-k} m_i \eta_i$$

$$\sum_{j-1}^{j-k} m_i \eta_i \equiv 0 \quad \text{if} \quad j-k > j-1$$

In Eq. (29b), negative values of the index k correspond to particles located in the plus η half of the η - ζ plane, and the converse is true for positive values of the index.

An examination of Eqs. (29a) and (29b) reveals a set of (n-1) coupled, non-linear homogeneous equations. It is immediately obvious that, for values of $\dot{\alpha}$ on the order of ω_c, the stability of the system of particles is questionable. For this condition the alternating tension and compression forces due to the gravitational gradient are predominant.

Two limiting cases may be analyzed by approximate methods. The first case is that in which the spin rate of the wire is very large compared to ω_c. In the second case the wire is almost aligned with the direction of the local vertical, and the angular rate with respect to the vertical is very small.

Case I

We will assume that $\dot{\alpha}$ is almost a constant.

$$\dot{\alpha} = \Omega + \Delta \dot{\alpha} \tag{30}$$

where

$$\Omega = \dot{\alpha}(o) \gg \Delta \dot{\alpha}$$

The parameters s_1 and s_2 may be written as follows:

$$s_1 \cong \Omega^2 + 2\,\Omega\,\Delta\dot{\alpha} + 2\,\Omega\,\omega_c + 2\omega_c\,\Delta\dot{\alpha} + \frac{3}{2}\,\omega_c^2\,(1 + \cos 2\alpha) \tag{31a}$$

$$s_2 \cong \Omega^2 + 2\,\Omega\,\Delta\dot{\alpha} + 2\,\Omega\,\omega_c + 2\omega_c\,\Delta\dot{\alpha} + \frac{3}{2}\,\omega_c^2\,(1 - \cos 2\alpha) \tag{31b}$$

For orbit altitudes ranging between 300 and 2000 miles, ω_c is of the order of 10^{-3} radians per second. With Ω approximately equal to one radian per second, s_1 and s_2 are almost equal to Ω^2 . Thus it might be expected that the form of the solution of Eq. (29b) is primarily oscillatory in nature. Since, in the derivation of the equations of motion, the ζ's have been assumed to be small, products such as $\zeta\,\Delta\dot{\alpha}$ will be neglected as higher order terms. With this assumption Eqs. (29a) and (29b) may be approximated as follows:

$$\Delta\dot{\alpha} + \frac{3}{2}\,\omega_c^2 \sin 2\alpha = \frac{(\zeta_{j-1} - \zeta_{j+1})}{2l_p}\left[\Omega^2 + 2\Omega\,\omega_c + \frac{3}{2}\,\omega_c^2\,(1 + \cos 2\alpha)\right] \tag{32a}$$

$$\ddot{\zeta}_{j\mp k} + \zeta_{j\mp k} \left\{ - \left[\Omega^2 + 2\,\Omega\,\omega_c + \frac{3}{2}\,\omega_c^2\,(1 - \cos 2\alpha) \right] + \left[\Omega^2 + 2\,\Omega\,\omega_c + \frac{3}{2}\,\omega_c^2\,(1 + \cos 2\alpha) \right] \left(\frac{2Q + m_{j-k}\,\eta_{j-k}}{m_{j-k}\,l_p} \right) \right\}$$

$$= \left[\Omega^2 + 2\,\Omega\,\omega_c + \frac{3}{2}\,\omega_c^2\,(1 + \cos 2\alpha) \right] \left[\frac{\eta_{j\mp k}}{2l_p}\,(\zeta_{j-1} - \zeta_{j+1}) + \frac{Q + m_{j-k}\,\eta_{j-k}}{m_{j-k}\,l_p}\,\zeta_{j\mp(k-1)} + \frac{Q\,\zeta_{j\mp(k+1)}}{m_{j-k}\,l_p} \right] \tag{32b}$$

Aside from the trigonometric terms, sin 2α and cos 2α, the coefficients of Eqs. (32a) and (32b) are constants. In view of the magnitude of ω_c^2, which multiplies these functions, a series solution in terms of a small parameter will be assumed. Thus

$$\Delta\dot{\alpha}(t,\epsilon) = \Delta\dot{\alpha}_{\epsilon=0} + \left(\frac{\partial \Delta\dot{\alpha}}{\partial \epsilon} \right)_{\epsilon=0} \epsilon + \ldots \tag{33a}$$

$$\zeta_{j\mp k}(t,\epsilon) = (\zeta_{j\mp k})_{\epsilon=0} + \left(\frac{\partial \zeta_{j\mp k}}{\partial \epsilon} \right)_{\epsilon=0} \epsilon + \ldots \tag{33b}$$

where

$$\epsilon = \frac{\omega_c^2}{\Omega^2 + 2\,\Omega\,\omega_c + \frac{3}{2}\,\omega_c^2} \equiv \frac{\omega_c^2}{\Omega_o^2}$$

With ϵ equal to zero Eq. (32b) is homogeneous, and may be solved in terms of trigonometric functions. After the ζ's with ϵ zero have been determined, $\Delta\dot{\alpha}_{\epsilon=0}$ can then be found from Eq. (32a). The perturbation differential equations are obtained by differentiating Eqs. (32a) and (32b) with respect to ϵ, and then setting ϵ equal to zero.

$$\Delta\ddot{\alpha}'_{\epsilon=o} + \frac{3}{2}\,\Omega_o^2 \sin 2\alpha_{\epsilon=o} = \frac{(\zeta_{j-1} - \zeta_{j+1})_{\epsilon=o}}{2\,l_p}\left[\frac{3}{2}\,\Omega_o^2 \cos 2\alpha_{\epsilon=o}\right] \tag{34a}$$

$$+ \frac{(\zeta'_{j-1} - \zeta'_{j+1})_{\epsilon=o}}{2\,l_p}\,\Omega_o^2$$

$$(\ddot{\zeta}_{j\mp k})_{\epsilon=o} + (\zeta'_{j\mp k})_{\epsilon=o}\left[-\Omega_o^2 + \Omega_o^2\left(\frac{2Q + m_{j-k}\eta_{j-k}}{m_{j-k}\,l_p}\right)\right] = \tag{34b}$$

$$- (\zeta_{j\mp k})_{\epsilon=o}\left[\frac{3}{2}\,\Omega_o^2 \cos 2\alpha_{\epsilon=o} + \frac{3}{2}\,\Omega_o^2 \cos 2\alpha_{\epsilon=o}\left(\frac{2Q + m_{j-k}\eta_{j-k}}{m_{j-k}l_p}\right)\right] + \frac{3}{2}\,\Omega_o^2 \cos 2\alpha_{\epsilon=o}\left[\frac{\eta_{j\mp k}}{2\,l_p}(\zeta_{j-1} - \zeta_{j+1})_{\epsilon=o}\right.$$

$$\left. + \frac{Q}{m_{j-k}\,l_p}\,\zeta_{j\mp(k+1)\epsilon=o} + \frac{Q + m_{j-k}\eta_{j-k}}{m_{j-k}\,l_p}\,\zeta_{j\mp(k-1)\epsilon=o}\right]$$

$$+ \Omega_o^2\left[\frac{\eta_{j\mp k}}{2\,l_p}(\zeta'_{j-1} - \zeta'_{j+1})_{\epsilon=o} + \frac{Q}{m_{j-k}\,l_p}\,\zeta'_{j\mp(k+1)\epsilon=o}\right.$$

$$\left. + \frac{Q + m_{j-k}\eta_{j-k}}{m_{j-k}l_p}\,\zeta'_{j\mp(k-1)\epsilon=o}\right]$$

where the prime denotes the operation $\frac{\partial}{\partial\epsilon}$.

An examination of Eqs. (34a) and (34b) reveals a set of non-homogeneous equations, with the forcing functions arising from the ϵ equal zero solutions. Higher-order perturbation equations may be obtained by continued differentiation of Eqs. (34a) and (34b) before setting ϵ equal to zero.

Case II

It is assumed that the wire is nearly aligned with the local vertical. Under these conditions both α and $\dot{\alpha}$ as well as the ζ's are small. With these approximations Eqs. (29a) and (29b) may be written as follows:

$$\ddot{\alpha} + 3\omega_c^2 \alpha = \frac{(\zeta_{j-1} - \zeta_{j+1})}{2 l_p} \Omega_1^2 \tag{35a}$$

$$\ddot{\zeta}_{j\mp k} + \zeta_{j\mp k} \left[- \Omega_2^2 + \Omega_1^2 \left(\frac{2Q + m_{j-k} \eta_{j-k}}{m_{j-k} l_p} \right) \right] =$$

$$\Omega_1^2 \left[\frac{\eta_{j\mp k}}{2 l_p} (\zeta_{j-1} - \zeta_{j+1}) + \frac{Q + m_{j-k}\eta_{j-k}}{m_{j-k} l_p} \zeta_{j\mp(k-1)} + \frac{Q \zeta_{j\mp(k+1)}}{m_{j-k} l_p} \right] \tag{35b}$$

where

$$\Omega_1^2 = \Omega^2 + 2 \Omega \omega_c + 3\omega_c^2$$

$$\Omega_2^2 = \Omega^2 + 2 \Omega \omega_c$$

For this case, the set of equations represented by Eq. (35b) may be solved immediately in terms of sinusoidal functions. Once the ζ's are known Eq. (35a) may be solved for α.

To illustrate the procedure described in the preceding paragraphs, a specific case will now be examined. Let the number of particles, n, be

5, then it follows that j is 3 and the index k is either zero or one. Upon consideration of Eqs. (29a) and (29b) and Eqs. (27a) and (27b), the following expressions may be deduced:

$$\ddot{\alpha} + 3\omega_c^2 \sin\alpha\cos\alpha = \frac{(\zeta_2 - \zeta_4)}{2 I_p} s_1 \tag{36a}$$

$$\ddot{\zeta}_2 + \zeta_2 \left[- s_2 + \frac{3}{2} s_1 + \left(\frac{4m_1 + m_2}{m_2}\right) s_1 \right] = s_1 \left[\zeta_3 \left(\frac{2m_1 + m_2 - m_3}{m_2}\right) - \frac{\zeta_4}{2} \right] \tag{36b}$$

$$\ddot{\zeta}_3 + \zeta_3 \left[- s_2 + \left(\frac{4m_1 + 2m_2}{m_3}\right) s_1 \right] = s_1 \left(\frac{2m_1 + m_2}{m_3}\right)(\zeta_2 + \zeta_4) \tag{36c}$$

$$\ddot{\zeta}_4 + \zeta_4 \left[- s_2 + \frac{3}{2} s_1 + \left(\frac{4m_1 + m_2}{m_2}\right) s_1 \right] = s_1 \left[\zeta_3 \left(\frac{2m_1 + m_2 - m_3}{m_2}\right) - \frac{\zeta_2}{2} \right] \tag{36d}$$

An inspection of Eqs. (36a-d) suggests that the following transformations would be useful:

$$\zeta_2 - \zeta_4 = x \tag{37a}$$

$$\zeta_2 + \zeta_4 = y \tag{37b}$$

Thus

$$\ddot{\alpha} + 3\omega_c^2 \sin\alpha\cos\alpha = \frac{s_1}{2 I_p} x \tag{38a}$$

$$\ddot{x} + x \left[s_1 - s_2 + \left(\frac{4m_1 + m_2}{m_2}\right) s_1 \right] = 0 \tag{38b}$$

$$\ddot{y} + y \left[2s_1 - s_2 + \left(\frac{4m_1 + m_2}{m_2}\right) s_1 \right] = 2s_1 \left(\frac{2m_1 + m_2 - m_3}{m_2}\right) \zeta_3 \tag{38c}$$

$$\ddot{\zeta}_3 + \zeta_3 \left[- s_2 + \left(\frac{4m_1 + 2m_2}{m_3}\right) s_1 \right] = s_1 \left(\frac{2m_1 + m_2}{m_3}\right) y \tag{38d}$$

We are now in a position to obtain the solutions of Eqs. (38a-d) for the two cases under consideration.

Case I

The first step is to express $\Delta\dot{\alpha}$, x, y and ζ_3 in a Taylor series in ϵ.

$$\Delta\dot{\alpha} = \Delta\dot{\alpha}_{\epsilon=0} + \left(\frac{\partial \Delta\dot{\alpha}}{\partial \epsilon}\right)_{\epsilon=0} \epsilon + \ldots \tag{39a}$$

$$x = x_{\epsilon=0} + \left(\frac{\partial x}{\partial \epsilon}\right)_{\epsilon=0} \epsilon + \ldots \tag{39b}$$

$$y = y_{\epsilon=0} + \left(\frac{\partial y}{\partial \epsilon}\right)_{\epsilon=0} \epsilon + \ldots \tag{39c}$$

$$\zeta_3 = (\zeta_3)_{\epsilon=0} + \left(\frac{\partial \zeta_3}{\partial \epsilon}\right)_{\epsilon=0} \epsilon + \ldots \tag{39d}$$

With ϵ equal to zero Eqs. (38b-d) may be solved immediately.

$$x_{\epsilon=0} = A_1 \cos \omega_x t + A_2 \sin \omega_x t \tag{40a}$$

$$y_{\epsilon=0} = B_1 \cos \omega_1 t + B_2 \sin \omega_1 t + B_3 \cos \omega_2 t + B_4 \sin \omega_2 t$$

(40b)

$$(\zeta_3)_{\epsilon=0} = C_1 \cos \omega_1 t + C_2 \sin \omega_1 t + C_3 \cos \omega_2 t + C_4 \sin \omega_2 t$$

(40c)

where

$$\omega_x^2 = \Omega_o^2 \left(\frac{4m_1 + m_2}{m_2}\right)$$

$$\omega_1^2 = \Omega_o^2 \left[\frac{4m_1 + 2m_2 - m_3}{2m_3} + \frac{4m_1 + 2m_2}{2m_2} - \sqrt{\left(\frac{4m_1 + 2m_2 - m_3}{2m_3}\right)^2 - \left(\frac{4m_1 + 2m_2}{2m_2}\right) + \left(\frac{4m_1 + 2m_2}{2m_2}\right)^2}\,\right]$$

$$\omega_2^2 = \Omega_o^2 \left[\frac{4m_1 + 2m_2 - m_3}{2m_3} + \frac{4m_1 + 2m_2}{2m_2} + \sqrt{\left(\frac{4m_1 + 2m_2 - m_3}{2m_3}\right)^2 - \left(\frac{4m_1 + 2m_2}{2m_2}\right) + \left(\frac{4m_1 + 2m_2}{2m_2}\right)^2}\,\right]$$

The A's, B's and C's are functions of the initial values of the ζ's and the $\dot{\zeta}$'s. For convenience it will be assumed that all of the $\dot{\zeta}$'s are initially zero. Under these conditions,

$$A_1 = x(o) = \zeta_2(o) - \zeta_4(o)$$

$$A_2 = 0$$

$$B_1 = \frac{y(o)\left[\omega_2^2 - \Omega_o^2\left(\frac{4m_1 + 2m_2}{m_2}\right)\right] + \zeta_3(o)\,\Omega_o^2\left[\frac{4m_1 + 2m_2 - 2m_3}{m_2}\right]}{\omega_2^2 - \omega_1^2}$$

$$B_2 = 0$$

$$B_3 = \frac{y(o)\left[\Omega_o^2\left(\frac{4m_1 + 2m_2}{m_2}\right) - \omega_1^2\right] - \zeta_3(o)\,\Omega_o^2\left[\frac{4m_1 + 2m_2 - 2m_3}{m_2}\right]}{\omega_2^2 - \omega_1^2}$$

$$B_4 = 0$$

$$C_1 = \frac{\zeta_3(o)\left[\omega_2^2 - \Omega_o^2\left(\frac{4m_1 + 2m_2 - m_3}{m_3}\right)\right] + y(o)\,\Omega_o^2\left(\frac{2m_1 + m_2}{m_3}\right)}{\omega_2^2 - \omega_1^2}$$

$$C_2 = 0$$

$$C_3 = \frac{\zeta_3(o)\left[\Omega_o^2\left(\frac{4m_1 + 2m_2 - m_3}{m_3}\right) - \omega_1^2\right] - y(o)\,\Omega_o^2\left(\frac{2m_1 + m_2}{m_3}\right)}{\omega_2^2 - \omega_1^2}$$

$$C_4 = 0$$

As anticipated, the basic solutions of Eqs. (38) are oscillatory in nature. Furthermore an examination of the expressions for ω_x, ω_1 and ω_2

indicates that for any mass distribution that is physically possible, the ϵ equal zero solutions are always composed of simple sinusoids.

With $x_{\epsilon=0}$ known it is now possible to find $\Delta\dot{\alpha}_{\epsilon=0}$ from Eq. (38a).

$$\Delta\dot{\alpha}_{\epsilon=0} = \frac{\Omega_o^2}{2l_p}\left[\frac{A_1 \sin \omega_x t}{\omega_x} - \frac{A_2 \cos \omega_x t}{\omega_x}\right] + \frac{A_2 \Omega_o^2}{2l_p \omega_x} \tag{41}$$

We can now check the validity of the assumption that $\Delta\dot{\alpha}$ is small and of the order of x. With A_2 zero

$$\Delta\dot{\alpha}_{\epsilon=0} = \frac{\Omega_o \, x(o) \sin \omega_x t}{2l_p\sqrt{\frac{4m_1 + m_2}{m_2}}}$$

Due to x(o), $\Delta\dot{\alpha}^2_{\epsilon=0}$ and $x_{\epsilon=0} \, \Delta\dot{\alpha}_{\epsilon=0}$ are second-order terms. For a wire 2000 ft long and with x(o) equal to 1 cm, the magnitude of $\Delta\dot{\alpha}_{\epsilon=0}$ is approximately 3×10^{-5} radians per second. Thus the retention of the ω_c^2 terms, in s_1 and s_2, as compared to the $\Omega\Delta\dot{\alpha}$ is justified only if x(o) is on the order of 10^{-3} cm.

The first-order perturbation equations are found by differentiating Eqs. (38a-d) with respect to ϵ, and then letting ϵ go to zero.

$$\Delta\ddot{\alpha}'_{\epsilon=0} + \frac{3}{2}\Omega_o^2 \sin 2\alpha_{\epsilon=0} = \frac{x_{\epsilon=0}}{2\,l_p}\left(\frac{3}{2}\Omega_o^2 \cos 2\alpha_{\epsilon=0}\right) + \frac{\Omega_o^2}{2\,l_p} x'_{\epsilon=0} \tag{42a}$$

$$\ddot{x}'_{\epsilon=0} + x'_{\epsilon=0}\,\omega_x^2 = - x_{\epsilon=0}\left[3\Omega_o^2 \cos 2\alpha_{\epsilon=0} + \left(\frac{4m_1 + m_2}{m_2}\right)\frac{3}{2}\Omega_o^2 \cos 2\alpha_{\epsilon=0}\right] \tag{42b}$$

$$\ddot{y}'_{\epsilon=0} + y'_{\epsilon=0}\,(\Omega_o^2 + \omega_x^2) \;=\; -\,y_{\epsilon=0}\left[\frac{9}{2}\,\Omega_o^2\,\cos 2\alpha_{\epsilon=0} + \left(\frac{4m_1 + m_2}{m_2}\right)\frac{3}{2}\,\Omega_o^2\,\cos 2\alpha_{\epsilon=0}\right]$$

$$+\;3\Omega_o^2\,\cos 2\alpha_{\epsilon=0}\left(\frac{2m_1 + m_2 - m_3}{m_2}\right)(\zeta_3)_{\epsilon=0} \qquad (42c)$$

$$+\;2\Omega_o^2\left(\frac{2m_1 + m_2 - m_3}{m_2}\right)(\zeta_3')_{\epsilon=0}$$

$$(\ddot{\zeta}_3')_{\epsilon=0} + (\zeta_3')_{\epsilon=0}\left[-\,\Omega_o^2 + \left(\frac{4m_1 + 2m_2}{m_3}\right)\Omega_o^2\right] \;=\; -\,(\zeta_3)_{\epsilon=0}\left[\frac{3}{2}\,\Omega_o^2\,\cos 2\alpha_{\epsilon=0}\right.$$

$$\left. +\left(\frac{4m_1 + 2m_2}{m_3}\right)\frac{3}{2}\,\Omega_o^2\,\cos 2\alpha_{\epsilon=0}\right] + \frac{3}{2}\,\Omega_o^2\,\cos 2\alpha_{\epsilon=0}\left(\frac{2m_1 + m_2}{m_3}\right)y_{\epsilon=0}$$

$$+\;\Omega_o^2\left(\frac{2m_1 + m_2}{m_3}\right)y'_{\epsilon=0} \qquad (42d)$$

Equations (42a-d) are linear, but non-homogeneous. Since all of the initial conditions have been satisfied by the ϵ zero solutions, the initial values of displacement and rate-of-change of displacement must be zero for all of the perturbation variables.

Upon substitution of Eqs. (40a-c) into Eqs. (42a-d) the solutions of Eqs. (42b-d) may be found

$$x'_{\epsilon=0} \;=\; \frac{x(o)\,(3\,\Omega_o^2 + \frac{3}{2}\,\omega_x^2)}{8\Omega(\omega_x + \Omega)(\omega_x - \Omega)}\left[2\Omega\cos\omega_x t + (\omega_x - \Omega)\cos(2\Omega + \omega_x)t\right.$$

$$- (\omega_x + \Omega) \cos (2\Omega - \omega_x)t \Big] \qquad (43a)$$

$$y'_{\epsilon=0} = B_1' \cos \omega_1 t + B_3' \cos \omega_2 t + K_1 \cos (2\Omega - \omega_x)t \qquad (43b)$$

$$+ K_2 \cos (2\Omega + \omega_1)t + K_3 \cos (2\Omega - \omega_2)t + K_4 \cos (2\Omega + \omega_2)t$$

$$(\zeta_3')_{\epsilon=0} = C_1' \cos \omega_1 t + C_3' \cos \omega_2 t + K_5 \cos (2\Omega - \omega_1)t \qquad (43c)$$

$$+ K_6 \cos (2\Omega + \omega_1)t + K_7 \cos (2\Omega - \omega_2)t + K_8 \cos (2\Omega + \omega_2)t$$

where

$$B_1' = \frac{1}{\omega_2^2 - \omega_1^2} \Big[3n_2 (C_1 + C_3) - n_1(B_1 + B_2) - \omega_2^2 (K_1 + K_2 + K_3 + K_4)$$

$$+ K_1(2\Omega - \omega_1)^2 + K_2(2\Omega + \omega_1)^2 + K_3(2\Omega - \omega_2)^2 + K_4(2\Omega + \omega_2)^2 \Big]$$

$$B_3' = \frac{1}{\omega_1^2 - \omega_2^2} \Big[3n_2 (C_1 + C_3) - n_1(B_1 + B_2) - \omega_1^2 (K_1 + K_2 + K_3 + K_4)$$

$$+ K_1(2\Omega - \omega_1)^2 + K_2 (2\Omega + \omega_1)^2 + K_3(2\Omega - \omega_2)^2 + K_4(2\Omega + \omega_2)^2 \Big]$$

$$C_1' = \frac{1}{\omega_2^2 - \omega_1^2} \Big[\frac{3}{2} n_4 (B_1 + B_3) - n_3(C_1 + C_3) - \omega_2^2 (K_5 + K_6 + K_7 + K_8)$$

$$+ K_5(2\Omega - \omega_1)^2 + K_2 (2\Omega + \omega_1)^2 + K_7 (2\Omega - \omega_2)^2 + K_8(2\Omega + \omega_2)^2 \Big]$$

$$C_3' = \frac{1}{\omega_1^2 - \omega_2^2}\left[\frac{3}{2} n_4 (B_1 + B_3) - n_3(C_1 + C_3) - \omega_1^2 (K_5 + K_6 + K_7 + K_8)\right.$$

$$\left. + K_5 (2\Omega - \omega_1)^2 + K_6(2\Omega + \omega_1)^2 + K_7(2\Omega - \omega_2)^2 + K_8(2\Omega + \omega_2)^2\right]$$

$$K_1 = \frac{3n_2 n_4 B_1 - 2n_2 n_3 C_1 + (3n_2 C_1 - n_1 B_1)\left[\left(\frac{4m_1 + 2m_2 - m_3}{m_3}\right)\Omega_o^2 - (2\Omega + \omega_1)^2\right]}{8\Omega (\Omega - \omega_1)(2\Omega - \omega_1 + \omega_2)(2\Omega - \omega_1 - \omega_2)}$$

$$K_2 = \frac{3n_2 n_4 B_1 - 2n_2 n_3 C_1 + (3n_2 C_1 - n_1 B_1)\left[\left(\frac{4m_1 + 2m_2 - m_3}{m_3}\right)\Omega_o^2 - (2\Omega + \omega_1)^2\right]}{8\Omega (\Omega + \omega_1)(2\Omega + \omega_1 + \omega_2)(2\Omega + \omega_1 - \omega_2)}$$

$$K_3 = \frac{3n_2 n_4 B_3 - 2n_2 n_3 C_3 + (3n_2 C_3 - n_1 B_3)\left[\left(\frac{4m_1 + 2m_2 - m_3}{m_3}\right)\Omega_o^2 - (2\Omega - \omega_2)^2\right]}{8\Omega (\Omega - \omega_2)(2\Omega - \omega_2 + \omega_1)(2\Omega - \omega_2 - \omega_1)}$$

$$K_4 = \frac{3n_2 n_4 B_3 - 2n_2 n_3 C_3 + (3n_2 C_3 - n_1 B_3)\left[\left(\frac{4m_1 + 2m_2 - m_3}{m_3}\right)\Omega_o^2 - (2\Omega + \omega_2)^2\right]}{8\Omega (\Omega + \omega_2)(2\Omega + \omega_2 + \omega_1)(2\Omega + \omega_2 - \omega_1)}$$

$$K_5 = \frac{3n_2 n_4 C_1 - n_1 n_4 B_1 + (\frac{3}{2} n_4 B_1 - n_3 C_1)\left[\Omega_o^2 + \omega_x^2 - (2\Omega - \omega_1)^2\right]}{8\Omega (\Omega - \omega_1)(2\Omega - \omega_1 + \omega_2)(2\Omega - \omega_1 - \omega_2)}$$

$$K_6 = \frac{3n_2 n_4 C_1 - n_1 n_4 B_1 + (\frac{3}{2} n_4 B_1 - n_3 C_1)\left[\Omega_o^2 + \omega_x^2 - (2\Omega + \omega_1)^2\right]}{8\Omega (\Omega + \omega_1)(2\Omega + \omega_1 + \omega_2)(2\Omega + \omega_1 - \omega_2)}$$

$$K_7 = \frac{3n_2\, n_4\, C_3 - n_1\, n_4\, B_3 + (\frac{3}{2}\, n_4\, B_3 - n_3\, C_3)\left[\Omega_o^2 + \omega_x^2 - (2\Omega - \omega_2)^2\right]}{8\Omega(\Omega - \omega_2)(2\Omega - \omega_2 + \omega_1)(2\Omega - \omega_2 - \omega_1)}$$

$$K_8 = \frac{3n_2\, n_4\, C_3 - n_1\, n_4\, B_3 + (\frac{3}{2}\, n_4\, B_3 - n_3\, C_3)\left[\Omega_o^2 + \omega_x^2 - (2\Omega + \omega_2)^2\right]}{8\Omega\ (\Omega + \omega_2)(2\Omega + \omega_2 + \omega_1)(2\Omega + \omega_2 - \omega_1)}$$

$$n_1 = 6\Omega_o^2\ (\frac{m_1 + m_2}{m_2})\ ; \qquad n_2 = \Omega_o^2\ (\frac{2m_1 + m_2 - m_3}{m_2})$$

$$n_3 = \frac{3}{2}\,\Omega_o^2\ (\frac{4m_1 + 2m_2 + m_3}{m_3})\ ; \qquad n_4 = \Omega_o^2\ (\frac{2m_1 + m_2}{m_3})$$

Finally, the perturbation spin rate may be found by introducing Eq. (43a) into Eq. (42a).

$$\Delta\dot{\alpha}_{\epsilon=o} = \frac{3}{4}\,\frac{\Omega_o^2}{\Omega}\ (\cos 2\Omega t - 1) - \frac{3}{4}\,\frac{\Omega_o^4\ x(o)}{1_p\ \omega_x^2\,\Omega}\ \sin 2\Omega t$$

$$+ \frac{\Omega_o^2\ x(o)}{4\ 1_p\ \omega_x^2}\ (\frac{3}{2}\,\omega_x^2 + 3\Omega_o^2)\ \left[\frac{\sin(2\Omega - \omega_x)t}{2\Omega - \omega_x} + \frac{\sin(2\Omega + \omega_x)t}{2\Omega + \omega_x}\right]$$

$$+ \frac{\Omega_o^2\ x(o)}{16\ 1_p\ \Omega}\,\frac{(\frac{3}{2}\,\omega_x^2 + 3\Omega_o^2)}{(\omega_x + \Omega)(\omega_x - \Omega)}\left[\frac{2\Omega\ \sin\omega_x t}{\omega_x} + \frac{(\omega_x - \Omega)\ \sin(2\Omega + \omega_x)t}{2\Omega + \omega_x}\right.$$

$$\left. - \frac{(\omega_x + \Omega)\ \sin\ (2\Omega - \omega_x)t}{2\Omega - \omega_x}\right] \qquad (44)$$

An examination of the first-order perturbation solutions indicates that a resonance condition exists if any one of the following conditions is met:

$$\omega_x - \Omega = 0 \qquad (a)$$

$$\omega_1 - \Omega = 0 \qquad (b)$$

$$\omega_2 - \Omega = 0 \qquad (c)$$

$$2\Omega - \omega_1 + \omega_2 = 0 \qquad (d) \qquad (45)$$

$$2\Omega - \omega_1 - \omega_2 = 0 \qquad (e)$$

$$2\Omega - \omega_2 + \omega_1 = 0 \qquad (f)$$

Case II

When the five-particle wire is aligned with the vertical, Eqs. (35a) and (35b) may be written as follows:

$$\ddot{\alpha} + 3\omega_c^2 \, \alpha = \frac{3\omega_c^2}{2\, l_p} \, x \qquad (46a)$$

$$\ddot{x} + 3\omega_c^2 \left(\frac{4m_1 + 2m_2}{m_2}\right) x = 0 \qquad (46b)$$

$$\ddot{y} + 3\omega_c^2 \left(\frac{4m_1 + 3m_2}{m_2}\right) y = 6\omega_c^2 \left(\frac{2m_1 + m_2 - m_3}{m_2}\right)\zeta_3 \qquad (46c)$$

$$\ddot{\zeta}_3 + 3\omega_c^2 \left(\frac{4m_1 + 2m_2}{m_3}\right) \zeta_3 = 3\omega_c^2 \left(\frac{2m_1 + m_2}{m_3}\right) y \tag{46d}$$

Equations (46a-d) are linear with constant coefficients, and thus may be solved in terms of exponential functions.

$$x = x(o) \cos \left(\omega_c \sqrt{3\left(\frac{4m_1 + 2m_2}{m_2}\right)} t \right) \tag{47a}$$

$$\alpha = \left[\alpha(o) + \frac{x(o)}{2l_p \left(\frac{4m_1}{m_2}\right)} \right] \cos \sqrt{3}\, \omega_c t$$

$$- \frac{x(o)}{2 l_p \left(\frac{4m_1}{m_2}\right)} \cos \left(\omega_c \sqrt{3\left(\frac{4m_1 + 2m_2}{m_2}\right)} t \right) \tag{47b}$$

$$y = B_1 \cos \omega_y t + B_3 \cos \omega_\zeta t \tag{47c}$$

$$\zeta_3 = C_1 \cos \omega_y t + C_3 \cos \omega_\zeta t \tag{47d}$$

where the B's and C's are functions of y(o) and ζ_3(o) and where

$$\omega_y^2 = 3\omega_c^2 \left[\frac{4m_1 + 2m_2}{2m_3} + \frac{4m_1 + 3m_2}{2m_2} + \sqrt{\left(\frac{4m_1 + 2m_2}{2m_3}\right)^2 + \left(\frac{4m_1 + 3m_2}{2m_2}\right)^2 - \left(\frac{4m_1 + 2m_2}{2m_2 m_3}\right)(m_2 + 2m_3)} \right]$$

$$\omega_\zeta^2 = 3\omega_c^2 \left[\frac{4m_1 + 2m_2}{2m_3} + \frac{4m_1 + 3m_2}{2m_2} - \sqrt{\left(\frac{4m_1 + 2m_2}{2m_3}\right)^2 + \left(\frac{4m_1 + 3m_2}{2m_2}\right)^2 - \frac{(4m_1 + 2m_2)(m_2 + 2m_3)}{2m_2\ m_3}} \right]$$

An examination of Eqs. (47 a-d) indicates that for small disturbances, the wire oscillates about the vertical. Without damping, these oscillations continue indefinitely, but the vertical configuration of the wire is completely stable.*

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*If the orbit has an eccentricity much greater than zero, the above conclusions are not valid.

IV. DISCUSSION OF THE RESULTS

The results of the previous section indicate that, for the spinning wire, parametric resonance can occur. In the example analyzed, only three normal modes of vibration are possible. (See Fig. 5) The center particle can oscillate at only two different frequencies, ω_1 and ω_2. The remaining particles can oscillate at three different frequencies, ω_x, ω_1 and ω_2.* Thus, because of the finite number of frequencies, resonance will occur only if one of the six possible combinations of the ω's with Ω happens to be exactly zero. However as the number of particles increases, the number of conditions under which resonance takes place also increases. Thus as n becomes infinite, resonance would surely occur, for the mathematical wire under study. The initial rotational kinetic energy would eventually be transferred to the various vibrational modes of the wire.

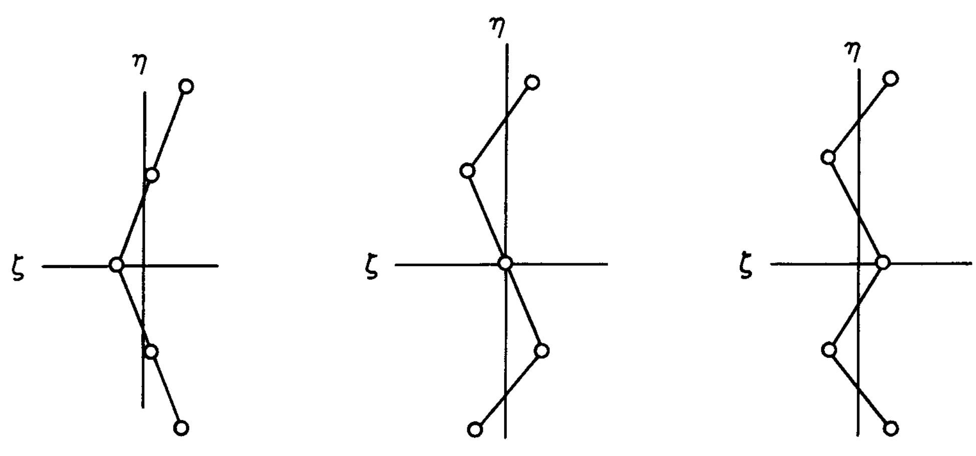

Fig. 5 — The normal vibrational modes for five particles

*All of the particles rotate with the η - ζ axes at the angular rate, $(\dot{\alpha} + \omega_c)$.

It is doubtful that a real wire would be rotationally unstable in the sense just described. In order to see this more clearly, let us review the physical and mathematical approximations that have been adopted in the development of the equations of motion.

Physical Approximations:

1. Assumption of a spherical gravitational potential function.
2. The neglect of internal restoring and dissipation forces, i.e., spring constant and damping terms.
3. The neglect of all external forces with the exception of number 1. Other forces which act on the wire include those due to interaction with the earth's magnetic field and electrostatic forces.
4. Restriction of the motion to a plane.

Mathematical Approximations:

1. The neglect of terms such as ζ_i^2, $\zeta_i, \dot{\zeta}_i$, $\zeta_i \, \Delta\alpha$, etc.
2. The neglect of terms of second degree and higher in the expansion of r_i^3.

Of the physical approximations, number 2 is probably the most significant from the viewpoint of stability. In particular, the addition of dissipative internal forces would modify the results of the preceding section. For a continuous wire, such damping forces arise due to internal friction between intergranular boundaries. An exact analysis of the problem is difficult due to the fact that the damping terms can be functions of both the amplitude and the rate of change of the ζ's. As a consequence it is even possible that the damping may be a destabilizing effect.*[5]

*In essence, the system might be self-excited due to an effective negative damping coefficient. The energy is supplied by the over-all rotational motion.

In the Appendix, the effect of viscous damping on the motion of the x normal mode is considered. It is shown that very small damping ratios, on the order of 0.05, are more than sufficient to eliminate any divergence problem. Furthermore, as long as the initial disturbances are such that the neglect of ζ^2 terms is valid, steady state conditions with essentially no transverse deflections are reached within a few minutes.

The neglect of spring-restoring forces is a reasonable approximation, since, for a flexible wire, they are very small compared with the tension restoring forces. If they were included in the analysis, the values of the undamped natural frequencies, ω_1, ω_2, etc., would be slightly increased.

Another physical approximation which might be important is number 3. A conducting wire rotating in the earth's magnetic field would be subjected to sinusoidally varying forces. An external forcing function of this type would be of importance if a resonance condition occurs.

Of the mathematical approximations, number 1 is probably the most important. The coupling between the vibrational and rotational degrees of freedom is nonlinear in nature. Thus the linearization procedure prior to the perturbation analysis eliminates the forced solution due to terms such as $x \Delta\dot{\alpha}$. It does not appear that the gross characteristics of the vibrational motion of the wire are altered by the neglect of such terms. However, when the forced motion of the wire with damping is considered, the energy dissipation of the system is strongly influenced by the nonlinear coupling terms. As a consequence, estimates of the lifetime of the spinning wire cannot be obtained from the linear model that has been examined.

The primary causes of forced motion are the interaction with the earth's magnetic field, mentioned above, and non-circular orbits.[3] Other external forcing functions would appear on the right-hand side of the equations of motion if higher-order terms are retained in the expansion of r_i^3, and if the earth's oblateness were considered.

The introduction of damping eliminates the possibility of either parametric or externally forced resonance. However, the continuous excitation of the wire due to such second-order effects as an oblate earth, a non-circular orbit, rotation of the wire out of the orbital plane, etc., will cause a gradual dissipation of the combined rotational and vibrational kinetic energy. Thus the spin rate of the wire will gradually decrease. When the tension forces due to spin are of the same order of magnitude as the gradient tension forces, the straight line configuration of the wire will no longer be stable.

Appendix

THE INFLUENCE OF VISCOUS DAMPING

As indicated in Section IV the internal dissipative forces of a wire cannot be represented by a simple viscous model. However, it might be possible to coat the wire with a substance which would impart the desired damping characteristics. In order to examine the influence of dissipative forces on the motion of the wire, it is assumed that a viscous damping process is applicable. From Fig. 2

$$F_{d_i} \sim \dot{\beta}_{i-1,i} - \dot{\beta}_{i,i+1} = \frac{1}{l_p}\left[\dot{\zeta}_{i-1,i} - 2\dot{\zeta}_i + \dot{\zeta}_{i+1}\right] \qquad (48)$$

If it is assumed that the damping is uniform per particle, then for the five-particle case

$$F_{d_1} = d(\dot{\zeta}_2 - \dot{\zeta}_1) \qquad (49a)$$

$$F_{d_2} = d(\dot{\zeta}_1 + \dot{\zeta}_3 - 2\dot{\zeta}_2) \qquad (49b)$$

$$F_{d_3} = d(\dot{\zeta}_2 + \dot{\zeta}_4 - 2\dot{\zeta}_3) \qquad (49c)$$

$$F_{d_4} = d(\dot{\zeta}_3 + \dot{\zeta}_5 - 2\dot{\zeta}_4) \qquad (49d)$$

$$F_{d_5} = d(\dot{\zeta}_4 - \dot{\zeta}_5) \qquad (49e)$$

Utilizing the center of mass relationships to eliminate $\dot{\zeta}_1$ and $\dot{\zeta}_5$, the equations of motion are

$$\ddot{\alpha} + \frac{3}{2}\,\omega_c^2 \sin 2\alpha = \frac{d\dot{x}}{2l_p(2m_1 + m_2)} + \frac{1}{2l_p}\,x\,s_1 \tag{50a}$$

$$\ddot{x} + d\left[\frac{(2m_1 + m_2)^2 - m_1 m_2}{m_1 m_2 (2m_1 + m_2)}\right]\dot{x} + x\left[s_1 - s_2 + \left(\frac{4m_1 + m_2}{m_2}\right) s_1\right] = 0 \tag{50b}$$

$$\ddot{y} + d\left[\frac{2m_1 + m_2}{m_1 m_2}\right]\dot{y} + y\left[2\,s_1 - s_2 + \left(\frac{4m_1 + m_2}{m_2}\right) s_1\right] =$$

$$\frac{d}{m_2}\left[\frac{2m_1 - m_3}{m_1}\right]\dot{\zeta}_3 + 2\,s_1\left(\frac{2m_1 + m_2 - m_3}{m_2}\right)\zeta_3 \tag{50c}$$

$$\ddot{\zeta}_3 + 2\,\frac{d}{m_3}\,\dot{\zeta}_3 + \zeta_3\left[-\,s_2 + \left(\frac{4m_1 + 2m_2}{m_3}\right) s_1\right] = \frac{d}{m_3}\,\dot{y} + s_1\left(\frac{2m_1 + m_2}{m_3}\right) y \tag{50d}$$

The solutions of Eqs. (50a-d) can be found by the methods employed in Section III. To illustrate the effect of damping, however, we will only consider Eqs. (50a) and (50b). With $\dot{x}(o) = 0$, the $\epsilon = o$ solution of Eq. (50b) is

$$x_{\epsilon=o} = x(o)e^{-\xi\omega_x t}\left(\cos \omega t + \xi\,\frac{\omega_x}{\omega}\sin \omega t\right) \tag{51}$$

where $\quad \omega = \omega_x\sqrt{1-\xi^2}$

$$2\xi\omega_x = \frac{d\left[(2m_1 + m_2)^2 - m_1 m_2\right]}{m_1 m_2 (2m_1 + m_2)}$$

If it is assumed that the damping ratio, ξ, is much less than one, then

$$\Delta\dot{\alpha}_{\epsilon=o} \cong \frac{\Omega_o^2\, x(o)}{1_p\, \omega_x}\left[\xi + \left(\frac{\omega}{2\,\omega_x}\sin\omega t - \xi\cos\omega t\right) e^{-\xi\omega_x t}\right] \tag{52}$$

$$+ \frac{dx(o)}{2\,\frac{1}{p}(2m_1 + m_2)}\left(e^{-\xi\omega_x t}\cos\omega t - 1\right)$$

The x perturbation equation of motion may be found by differentiating Eq. (50b) with respect to ϵ and then setting ϵ equal to zero.

$$\ddot{x}'_{\epsilon=o} + 2\xi\omega_x\,\dot{x}'_{\epsilon=o} + \omega_x^2\,x'_{\epsilon=o} = -\left(3\Omega_o^2 + \frac{3}{2}\,\omega_x^2\right) x(o)\,e^{-\xi\omega_x t}\left[\cos\omega t\cos 2\Omega t + \frac{\xi\omega_x}{\omega}\sin\omega t\cos 2\Omega t\right] \tag{53}$$

The solution of Eq. (53) is

$$x'_{\epsilon=o} = e^{-\xi\omega_x t}\left\{A_1'\cos\omega t + A_2'\sin\omega t - \frac{x(o)\left(3\Omega_o^2 + \frac{3}{2}\,\omega_x^2\right)}{8\Omega}\left[\frac{\cos(2\Omega - \omega)t}{(\omega - \Omega)} - \frac{\cos(2\Omega + \omega)t}{(\omega + \Omega)} - \frac{\xi\omega_x}{\omega}\,\frac{\sin(\omega + 2\Omega)t}{(\omega + \Omega)} + \frac{\xi\omega_x}{\omega}\,\frac{\sin(\omega - 2\Omega)t}{(\omega - \Omega)}\right]\right\} \tag{54}$$

With $x'_{\epsilon=o}(o) = x'_{\epsilon=o}(o) = 0$, Eq. (54) becomes

$$x'_{\epsilon=o} = e^{-\xi\omega_x t} \frac{x(o)(3\Omega_o^2 + \frac{3}{2}\omega_x^2)}{8\Omega(\omega-\Omega)(\omega+\Omega)} \Big[2\Omega \cos \omega t - (\omega+\Omega) \cos (2\Omega-\omega)t$$

$$+ (\omega-\Omega) \cos (2\Omega+\omega)t - 2\Omega \frac{\xi\omega_x}{\omega} \sin \omega t \tag{55}$$

$$+ (\omega-\Omega) \frac{\xi\omega_x}{\omega} \sin (\omega + 2\Omega)t - (\omega+\Omega) \frac{\xi\omega_x}{\omega} \sin (\omega-2\Omega)t \Big]$$

As Ω approaches ω, resonance occurs. Thus

$$x'_{\epsilon=o} = \frac{x(o)(3\Omega_o^2 + \frac{3}{2}\omega_x^2)}{16\,\omega^2} e^{-\xi\omega_x t} \Big[\cos 3\,\omega t - \cos \omega t - 4\omega t \sin \omega t$$

$$+ \frac{\xi\omega_x}{\omega} (\sin \omega t + \sin 3\,\omega t) - 4\xi\omega_x t \cos \omega t \Big] \tag{56}$$

The total solution for the normal vibrational mode, x, with viscous damping included, is:

$$x = x(o)\, e^{-\xi\omega_x t} \Big\{ \cos \omega t + \xi \frac{\omega_x}{\omega} \sin \omega t + \frac{\omega_c^2 \,(3\Omega_o^2 + \frac{3}{2}\omega_x^2)}{16\,\omega^2 \Omega_o^2} \Big[\cos 3\,\omega t$$

$$- \cos \omega t - 4\,\omega t \sin \omega t + \xi \frac{\omega_x}{\omega} (\sin \omega t + \sin 3\,\omega t)$$

$$- 4\xi\, \omega_x t \cos \omega t \Big] \Big\} \tag{57}$$

If circular orbits with altitudes of the order of 200 to 2000 miles are employed, then, without damping, the secular term of Eq. (57) predominates after a period of ten days to two weeks. However even with a value of ξ of the order of 0.05, the amplitude of x is 1/e of x(o) at the end of 20 seconds. The secular terms are completely dominated by the exponential after, approximately, 100 seconds.

Differentiation with respect to ϵ of Eq. (50a) yields the α perturbation equation.

$$\begin{aligned} \Delta\ddot{\alpha}'_{\epsilon=o} &= \frac{x_{\epsilon=o}}{2\,l_p}\left(3/2\,\Omega_o^2 \cos 2\Omega t\right) + \frac{\Omega_o^2}{2\,l_p}\,x'_{\epsilon=o} \\ &\quad + \frac{d\,\dot{x}'_{\epsilon=o}}{l_p(4m_1 + 2m_2)} - \frac{3}{2}\,\Omega_o^2 \sin\left(2\Omega t + 2\,\Delta\alpha_{\epsilon=o}\right) \end{aligned} \tag{58}$$

With the substitution of $x_{\epsilon=o}$, $x'_{\epsilon=o}$, $\dot{x}'_{\epsilon=o}$ and $\Delta\alpha_{\epsilon=o}$ into Eq. (58), $\Delta\dot{\alpha}'_{\epsilon=o}$ may be found by a direct integration. The resulting expression is quite complex, but it can be easily shown that all of the terms quickly damp to zero with one exception. This particular term would exist even if the body was rigid since it arises from the gravitational gradient torque term of Eq. (58). Physically the effect may be explained as follows: As the wire rotates about its center of mass, the potential energy of the wire changes periodically. The change in potential energy is reflected in a change of the rotational kinetic energy of the wire. Thus the spin rate also varies periodically.* This variation in the spin rate is extremely small, and it has no effect upon the vibrational motion until terms of the order of ϵ^2 are retained.

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*An exact analysis of this effect requires that the coupling between the rotational motion and the orbital motion be included as indicated in Ref. 3.

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